ON THE INTEGERS NOT OF THE FORM $p + 2^a + 2^b$

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Abstract. We prove that

$$|\{1 \leqslant n \leqslant x : n \text{ is odd and not of the form } p+2^a+2^b\}| \gg x \cdot \exp\left(-C \log x \cdot \frac{\log\log\log\log x}{\log\log\log x}\right),$$

where $C > 0$ is an absolute constant.

1. Introduction

As early as 1849, Polignac conjectured that every odd integer greater than 3 is the sum of a prime and a power of 2. Of course, Polignac's conjecture is not true, since 127 is an evident counterexample. In 1934, Romanoff [11] proved that the sumset

$$\{p + 2^b : p \text{ is prime},\ b \in \mathbb{N}\}$$

has a positive lower density. And in the other direction, van der Corput [2] proved that the set

$$\{n \geqslant 1 : n \text{ is odd and not of the form } p + 2^b\}$$

also has a positive lower density. In fact, with help of a covering system, Erdős [4] found that every positive integer n with $n \equiv 7629217 \pmod{11184810}$ is not representable as the sum of a prime and a power of 2.

In [3], Crocker proved that there exist infinitely many odd positive integers x not of the form $p+2^a+a^b$. One key of Crocker's proof is the following observation: If $b-a = 2^s t$ with $s \geqslant 0$ and $2 \nmid t$, then $2^a + 2^b \equiv 0 \pmod{2^{2^s} + 1}$. And Crocker also constructed a suitable covering system to deal with the case $a = b$. In [13], Sun and Le discussed the integers not of the form $p^\alpha + c(2^a + 2^b)$. And subsequently, Yuan [15] proved the there exist infinitely many positive odd integers x not of the form $p^\alpha + c(2^a + 2^b)$.

Let

$$\mathcal{N} = \{n \geqslant 1 : n \text{ is odd and not of the form } p + 2^a + 2^b\}$$

and

$$\mathcal{N}_* = \{n \geqslant 1 : n \text{ is odd and not of the form } p^\alpha + 2^a + 2^b\}.$$

Erdős asked whether $|\mathcal{N} \cap [1, x]| \gg x^\epsilon$ for some $\epsilon > 0$. And as Granville and Soundararajan [6] mentioned, this is true under the assumption that there exist infinitely many $m_1 < m_2 < m_3 < \ldots$ satisfying all $2^{2^{m_i}} + 1$ are composite and $\{m_{i+1} - m_i\}$ is bounded.

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Erdős even suggested [7, A19] that $|\mathcal{N}\cap[1,x]|\geqslant Cx$ for a constant $C>0$, though it seems that the covering congruences could not help here. In [1], Chen, Feng and Templier proved that

$$\limsup_{x\to\infty}\frac{|\mathcal{N}_*\cap[1,x]|}{x^{1/4}}=+\infty$$

if there exist infinitely many m satisfying $2^{2^m}+1$ is composite, and

$$\limsup_{x\to\infty}\frac{|\mathcal{N}_*\cap[1,x]|}{\sqrt{x}}>0$$

if there are only finite many m satisfying $2^{2^m}+1$ is prime. Recently, in his answer to a conjecture of Sun, Poonen [10] gave a heuristic argument which suggests that for each odd $k>0$,

$$|\{1\leqslant n\leqslant x:\, n \text{ is odd and not of the form } p+2^a+k\cdot 2^b\}|\gg_\epsilon x^{1-\epsilon}$$

for any $\epsilon>0$, where $\gg_\epsilon$ means the implied constant only depends on ϵ.

On the other hand, using Selberg's sieve method, Tao [14] proved that for any $\mathcal{K}\geqslant 1$ and sufficiently large x, the number of primes $p\leqslant x$ such that $|kp\pm ja^i|$ is composite for all $1\leqslant a,j,k\leqslant\mathcal{K}$ and $1\leqslant i\leqslant\mathcal{K}\log x$, is at least $C_{\mathcal{K}}x/\log x$, where $C_{\mathcal{K}}$ is a constant only depending on $\mathcal{K}$. Motivated by Tao's idea, in this short note, we shall unconditionally prove that

Theorem 1.1.

$$|\mathcal{N}_*\cap[1,x]|\gg x\cdot\exp\bigg(-C\log x\cdot\frac{\log\log\log\log x}{\log\log\log x}\bigg),$$

where $C>0$ is an absolute constant.

Clearly Theorem 1.1 implies $|\mathcal{N}_*\cap[1,x]|\gg_\epsilon x^{1-\epsilon}$ for any $\epsilon>0$. The proof of Theorem 1.1 will be given in the next section. And unless indicated otherwise, the constants implied by $\ll$, $\gg$ and $O(\cdot)$ are always absolute.

2. Proof of Theorem 1.1

Since

$$|\{1\leqslant n\leqslant x:\, n \text{ is of the form } p^\alpha+2^a+2^b \text{ with } \alpha\geqslant 2\}|=O(\sqrt{x}(\log x)^3),$$

we only need to show that

$$|\mathcal{N}\cap[1,x]|\gg x\cdot\exp\bigg(-C\log x\cdot\frac{\log\log\log\log x}{\log\log\log x}\bigg).$$

The following two lemmas are easy applications of the Selberg sieve method (cf. [8, Theorems 3.2 and 4.1], [9, Theorem 7.1]).

Lemma 2.1. *Suppose that $W \geqslant 1$ and β are integers with $(\beta, W) = 1$. Then*

$$|\{1 \leqslant n \leqslant x : Wn + \beta \text{ is prime}\}| \leqslant \frac{C_1 x}{\log x} \prod_{p|W} \left(1 - \frac{1}{p}\right)^{-1},$$

where C_1 is an absolute constant.

Lemma 2.2. *Suppose that x is a sufficiently large integer. Suppose that $p_1, p_2, \ldots, p_h$ are distinct primes less than $x^{\frac{1}{8}}$. Then*

$$|\{1 \leqslant n \leqslant x : n \not\equiv 0 \pmod{p_j} \text{ for every } 1 \leqslant j \leqslant h\}| \leqslant C_2 x \prod_{j=1}^{h} \left(1 - \frac{1}{p_j}\right),$$

where C_2 is an absolute constant.

The following lemma is due to Ford, Luca and Shparlinski [5, Theorem 1].

Lemma 2.3. *The series*

$$\sum_{n=1}^{\infty} \frac{(\log n)^{\gamma}}{P(2^n - 1)}$$

converges for any $\gamma < 1/2$, where $P(n)$ denotes the largest prime factor of n.

Let

$$C_3 = \sum_{p \text{ prime}} \frac{1}{P(2^p - 1)}.$$

Suppose that x is sufficiently large. Let

$$K = \left\lfloor \frac{\log \log \log x}{100 \log \log \log \log x} \right\rfloor$$

and $L = \log(2^9 C_1 C_2 K) + 2C_3$, where $\lfloor \theta \rfloor = \max\{z \in \mathbb{Z} : z \leqslant \theta\}$.

Let $u = e^{e^{K(L+1)}}$. By the Mertens theorem (cf. [9, Theorem 6.7]), we know that

$$\sum_{\substack{p \leqslant u \\ p \text{ prime}}} \frac{1}{p} = \log \log u + B + O\left(\frac{1}{\log u}\right) = K(L+1) + O(1).$$

where $B = 0.2614972\ldots$ is a constant. So we may choose some distinct odd primes less than u

$$p_{1,1}, \ldots, p_{1,h_1}; p_{2,1}, \ldots, p_{2,h_2}; \ldots; p_{K,1}, \ldots, p_{K,h_K}$$

such that

$$\sum_{j=1}^{h_i} \frac{1}{p_{i,j}} \geqslant L$$

for $1 \leqslant i \leqslant K$. Let $q_{i,j} = P(2^{p_{i,j}} - 1)$ for $1 \leqslant i \leqslant K$ and $1 \leqslant j \leqslant h_i$. Clearly these $q_{i,j}$ are all distinct. Now,

$$\sum_{j=1}^{h_i} \log\left(1 - \frac{1}{p_{i,j}}\right) \leqslant -\sum_{j=1}^{h_i} \frac{1}{p_{i,j}},$$

whence

$$\prod_{j=1}^{h_i} \left(1 - \frac{1}{p_{i,j}}\right) \leqslant e^{-L}.$$

And

$$\prod_{i=1}^{K}\prod_{j=1}^{h_i}\left(1 - \frac{1}{q_{i,j}}\right)^{-1} \leqslant \prod_{i=1}^{K}\prod_{j=1}^{h_i}\left(1 + \frac{2}{q_{i,j}}\right) \leqslant \left(\frac{\sum_{i=1}^{K}\sum_{j=1}^{h_i}(1 + 2/q_{i,j})}{\sum_{i=1}^{K} h_i}\right)^{\sum_{i=1}^{K} h_i} \leqslant e^{2C_3}.$$

Let

$$W_{1,i} = \prod_{j=1}^{h_i} q_{i,j}$$

for $1 \leqslant i \leqslant K$, and let

$$W_1 = \prod_{i=1}^{K} W_{1,i}.$$

Then

$$W_1 \leqslant 2^{\sum_{i=1}^{K}\sum_{j=1}^{h_i} p_{i,j}} \leqslant 2^{\frac{u^2}{\log u}},$$

since (cf. [12])

$$\sum_{\substack{p \leqslant u \\ p \text{ prime}}} p = \left(\frac{1}{2} + o(1)\right)\frac{u^2}{\log u}.$$

And noting that for sufficiently large x

$$\frac{\log\log\log(2^{\frac{u^2}{\log u}})}{\log\log\log(x^{\frac{1}{K}})} \leqslant \frac{2K(L+1)}{\log(\log\log x - \log K)} \leqslant 1,$$

we have $W_1 \leqslant x^{\frac{1}{K}}$.

Let $m = \lfloor \log_2 \log_2(x^{\frac{2}{K-1}}) \rfloor$ and $K' = 1 + \lfloor 2^{-m} \log_2 x \rfloor$, where $\log_2 x = \log x / \log 2$. We have

$$K' \leqslant 1 + \frac{\log_2 x}{2^m} \leqslant 1 + \frac{2\log_2 x}{2^{\log_2 \log_2(x^{\frac{2}{K-1}})}} = 1 + \frac{2\log_2 x}{\frac{2}{K-1} \cdot \log_2 x} = K.$$

For each $k \geqslant 0$, let γ_k be the smallest prime factor of $2^{2^k} + 1$. Let

$$W_2 = \prod_{k=0}^{m-1} \gamma_k$$

and $W = W_1W_2$. It is not difficult to see that $(W_1, W_2) = 1$. And

$$W \leqslant W_1 \prod_{k=0}^{m-1}(1 + 2^{2^k}) \leqslant x^{\frac{1}{K}} \cdot x^{\frac{2}{K-1}} \leqslant x^{\frac{3}{K-1}}.$$

Let β be an odd integer such that

$$\beta \equiv 2^{2^m(i-1)} + 1 \ (\mathrm{mod}\ \prod_{j=1}^{h_i} q_{i,j})$$

and

$$\beta \equiv 0 \ (\mathrm{mod}\ \gamma_k)$$

for $1 \leqslant i \leqslant K'$ and $0 \leqslant k \leqslant m-1$.

Let

$$\mathcal{S} = \{1 \leqslant n \leqslant x : \ n \equiv \beta \ (\mathrm{mod}\ 2W)\}.$$

Clearly,

$$\frac{x}{2W} - 1 \leqslant |\mathcal{S}| \leqslant \frac{x}{2W}.$$

Let

$$\mathcal{T}_1 = \{n \in \mathcal{S} : \ n \text{ is of the form } p + 2^a + 2^b \text{ with } p \mid W\}$$

and

$$\mathcal{T}_2 = \{n \in \mathcal{S} \setminus \mathcal{T}_1 : \ n \text{ is of the form } p + 2^a + 2^b \text{ with } p \nmid W\}.$$

Clearly $|\mathcal{T}_1| = O(W(\log x)^2)$.

Suppose that $n \in \mathcal{S}$ and $n = p + 2^a + 2^b$ with p is prime and $0 \leqslant a \leqslant b$. If $a \not\equiv b \ (\mathrm{mod}\ 2^m)$, then $b = a + 2^s t$ where $0 \leqslant s \leqslant m-1$ and $2 \nmid t$. Thus

$$p = n - 2^a(2^{2^s t} + 1) \equiv \beta - 2^a(2^{2^s} + 1)\sum_{j=0}^{t-1}(-1)^j 2^{2^s j} \equiv 0 \ (\mathrm{mod}\ \gamma_s).$$

Since p is prime, we must have $p = \gamma_s$, i.e., $n \in \mathcal{T}_1$.

Below we assume that $a \equiv b \ (\mathrm{mod}\ 2^m)$. Write $b - a = 2^m(t-1)$ where $1 \leqslant t \leqslant K'$. If $a \equiv 0 \ (\mathrm{mod}\ p_{t,j})$ for some $1 \leqslant j \leqslant h_t$, then recalling $2^{p_{t,j}} \equiv 1 \ (\mathrm{mod}\ q_{t,j})$, we have

$$p = n - 2^a(2^{2^m(t-1)} + 1) \equiv \beta - (2^{2^m(t-1)} + 1) \equiv 0 \ (\mathrm{mod}\ q_{t,j}).$$

So $p = q_{t,j}$ and $n \in \mathcal{T}_1$. On the other hand, for any $a \geqslant 0$ satisfying $a \not\equiv 0 \ (\mathrm{mod}\ p_{t,j})$ for all $1 \leqslant j \leqslant h_t$, i.e., $(a, W_{1,t}) = 1$, by Lemma 2.1, we have

$$\begin{aligned}&|\{n \in \mathcal{S} : \ n - 2^a(2^{2^m(t-1)} + 1) \text{ is prime}\}|\\ \leqslant&\frac{2C_1|\mathcal{S}|}{\log|\mathcal{S}|}\prod_{k=0}^{m-1}\left(1 - \frac{1}{\gamma_k}\right)^{-1}\prod_{i=1}^{K}\prod_{j=1}^{h_i}\left(1 - \frac{1}{q_{i,j}}\right)^{-1} \leqslant \frac{2^5C_1e^{2C_3}}{W} \cdot \frac{x}{\log x}\end{aligned}$$

since $\gamma_k \equiv 1 \pmod{2^{k+1}}$ and $\gamma_k > 2^{k+1}$. And noting that

$$\frac{\log\log u}{\log\log((\log_2 x)^{\frac{1}{8}})} \leqslant \frac{K(L+1)}{\log(\log\log x - \log\log 2 - \log 8)} < 1,$$

we have $u < (\log_2 x)^{\frac{1}{8}}$. By Lemma 2.2,

$$\begin{aligned}&|\{0 \leqslant a \leqslant \log_2 x : \ a \not\equiv 0 \pmod{p_{t,j}} \text{ for all } 1 \leqslant j \leqslant h_t\}|\\ \leqslant &C_2 \frac{\log x}{\log 2} \prod_{j=1}^{h_t} \left(1 - \frac{1}{p_{t,j}}\right) \leqslant 2C_2 e^{-L} \log x.\end{aligned}$$

Thus

$$\begin{aligned}|\mathcal{T}_2| \leqslant &\sum_{t=1}^{K'} \sum_{\substack{0 \leqslant a \leqslant \log_2 x \\ (a, W_{1,t})=1}} |\{n \in \mathcal{S} : \ n - 2^a(2^{2^m(t-1)} + 1) \text{ is prime}\}|\\ \leqslant &K \cdot \frac{2^5 C_1 e^{2C_3}}{W} \cdot \frac{x}{\log x} \cdot 2C_2 e^{-L} \log x \leqslant \frac{x}{4W}.\end{aligned}$$

It follows that

$$\begin{aligned}&|\{n \in \mathcal{S} : \ n \text{ is not of the form } p + 2^a + 2^b\}|\\ =&|\mathcal{S}| - |\mathcal{T}_1| - |\mathcal{T}_2| \geqslant \frac{x}{2W} - 1 - O(W(\log x)^2) - \frac{x}{4W} \gg x^{1-\frac{4}{K}}.\end{aligned}$$

The proof of Theorem 1.1 is complete. $\square$

Remark. Using a similar discussion, it is not difficult to prove that for any given $\mathcal{K} \geqslant 1$,

$$\begin{aligned}&|\{1 \leqslant n \leqslant x : \ n \text{ is odd and } n \neq p + c(2^a + 2^b) \text{ with } p \text{ prime}, \ a, b \geqslant 0, \ 1 \leqslant c \leqslant \mathcal{K}\}|\\ \gg_{\mathcal{K}} &x \cdot \exp\left(- C_{\mathcal{K}} \log x \cdot \frac{\log\log\log\log x}{\log\log\log x}\right),\end{aligned}$$

where the constant $C_{\mathcal{K}} > 0$ only depends on $\mathcal{K}$.

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References

[1] Y.-G. Chen, R. Feng and N. Templier, *Fermat numbers and integers of the form $a^k + a^l + p^\alpha$*, Acta Arith., **135**(2008), 51-61.

[2] J. G. van der Corput, *On de Polignacs conjecture*, Simon Stevin, **27**(1950), 99-105.

[3] R. Crocker, *On the sum of a prime and two powers of two*, Pacific J. Math., **36**(1971), 103-107.

[4] P. Erdős, *On integers of the form $2^k + p$ and some related problems*, Summa Brasil. Math., **2**(1950), 113-123.

[5] K. Ford, F. Luca and I. E. Shparlinski, *On the largest prime factor of the Mersenne numbers*, Bull. Austr. Math. Soc., **79** (2009), 455-463.

[6] A. Granville and K. Soundararajan, *A binary additive problem of Erdős and the order of* 2 mod p^2, Ramanujan J., **2**(1998), 283-298.
[7] R. K. Guy, *Unsolved problems in number theory*, 3rd ed., New York, Springer-Verlag, 2004.
[8] H. Halberstam and H.-E. Richert, *Sieve methods*, London Mathematical Society Monographs, **4**, Academic Press, London-New York, 1974.
[9] M. B. Nathanson, *Additive number theory. The classical bases*, Graduate Texts in Mathematics, **164**, Springer-Verlag, New York, 1996.
[10] B. Poonen, The 47 conjecture, 2009. http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0903&L=nmbrthry&T=0&P=1147
[11] N. Romanoff, *Über einige Sätze der additiven Zahlentheorie*, Math. Ann., **109**(1934), 668-678.
[12] T. Salát and S. Znám,*On sums of the prime powers*, Acta Fac. Rer. Nat. Univ. Comenianae, Math., **21**(1968), 21-25.
[13] Z.-W. Sun and M.-H. Le, *Integers not of the form* $c(2^a + 2^b) + p^\alpha$, Acta Arith., **99**(2001), 183-190.
[14] T. Tao, *A remark on primality testing and decimal expansions*, J. Austr. Math. Soc., to appear.
[15] P. Z. Yuan, *Integers not of the form* $c(2^a + 2^b) + p^\alpha$, Acta Arith., **115**(2004), 23-28.

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